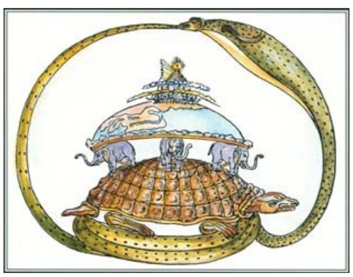
Les formes qui se déforment, la topologie *Vicente Muñoz* RBA, 2013, (175 p.), ISBN 978-2-8152-0476-7, hbk.

Vicente Muñoz

Since physicists started looking for a unifying theory, we know that our world needs more than the three dimensions that were needed to describe space. Since Einstein introduced space-time, we also know that the metric need not be positive definite, and with strings and M-theory, we are up to 11 and more dimensions. But how could to find out what kind of geometry we should use just thinking of the space we live in and what is the shape and the future of our universe?

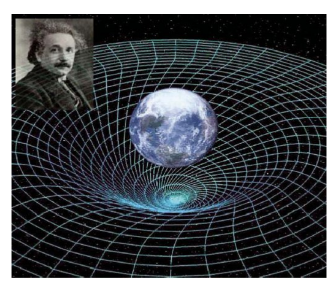
Vicente Muñoz, has agreed with the publisher of *Le Monde* to prepare a volume in their series *Le monde est mathématique* with the purpose to give the non-mathematician a glimpse on how one could conceive the shape of our spatial universe. The answer is approached from a topological-geometrical point of view. So there is no in depth fiddling with physics, strings or membranes to confuse the reader. Limited by our senses, we experience locally our environment as a Euclidean three-dimensional space meaning that we live on a manifold. But there is local and there is local at a cosmic scale. So perhaps it is not Euclidean in a global picture. Unfortunately, it is difficult for us to "visualize" a global higher-dimensional picture of the world we live in. Things we are not able to explain may be obvious for a higher-dimensional creature observing us from its hyperspace.



Chinese view:
giant turtle carries the world

E. Abbott Abbott in his novella *Flatland* (1884) has evocated what this higher-dimensional creature would experience observing our helpless discussions from its point of view. By reducing the dimensions, Abbott places the reader in the hyperspace of *Flatland*. He describes how flat geometrical creatures would think while they live in a 2D plane, unaware of the ambient 3D world. Besides this geometrical insight, the novella was also a satire of society at the end of the 19th century. A. Square, living in this 2D world succeeds in thinking "out of the plane", thereby using models of a lower-dimensional *Lineland* and even *Pointland*. He is able to grasp an external *Spaceland*'s view of his world inspired by a sphere visiting *Flatland*. It doesn't end well for A. Square, but that is a different non-mathematical lesson that Abbott wants to teach us.

Muñoz has picked up Abbott's idea and tells us how *Carrée* (A. Square) who is living in *Plateville* in *Flatland* would try to answer the questions formulated above for his 2D world. Helped by his compatriots he is able, using local geometrical experiments, to find out about the global topology of their world. That leaves them with only a few possibilities. Then, by analogy, similar ideas are lifted to our 3D experiences in the second part of the book.

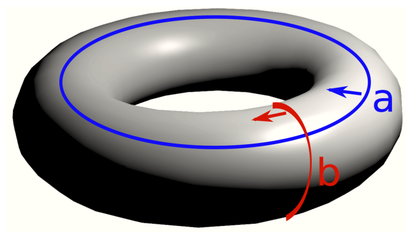


Einstein's curved space

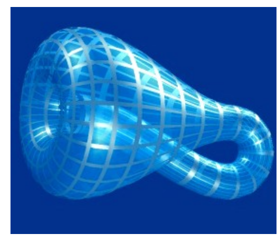
In a first chapter we are reminded of historical discussions about the form of planet earth, which has now eventually been recognized as being approximately spherical. But what is the form of the universe? We live on a 3-manifold in a higher-dimensional space, but what is its global form? What topology does it have? We usually agree on the hypothesis that the universe has no limit (there is no boundary). *Carrée*, living on the surface of a sphere does not experience a boundary, yet a sphere is compact. So what about the topology our universe? Another hypothesis, based on Big Bang theory is that the universe is homogeneous. Matter is distributed approximately uniformly. And this has also an influence on its geometry. Thus it is a mathematical challenge to find and classify all varieties that have these three properties: no boundary, compact, and homogeneous. How would *Carrée* solve this in 2D?

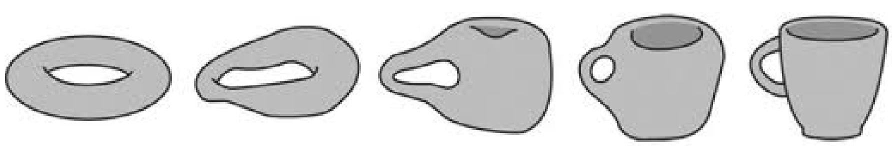
So suppose *Carrée* tries to explore his spherical world and, starting in a point A, follows a straight path North (a meridian in his world). Then he will pass through the North Pole, through the antipode of A, the South Pole and end up in A. No boundary was met. Repeating the experiment, starting in B, East of A, then the two paths will intersect in exactly two points (North and South Pole). The same will happen for any two different straight lines (i.e., great circles which are the geodesics on a sphere). There are always two points of intersection. There are no parallel lines. Suppose that *Carrée* and *Pentagone* both start at point A in different directions following a straight line, but that except for point A, there has been no other spot they both visited on their tour around the world, then their world can not be spherical. There must be a 'hole' in the manifold and their world is not a sphere, but a torus, or it might even be a more complicated form with more than one hole. After all his traveling *Carrée* can produce an atlas, consisting of pages that reflect the local geometry and that link to each other (leaving page x at the top means entering page y from the bottom etc.). A Möbius band world is one-sided and would only be possible if a globe-trotter would return to his starting point as a geometrical inverse of himself, but that idea is abandoned and considered to be the product of the imagination of 2D science fiction authors.

The next chapter introduces some topological definitions such as orientability, bottle of Klein, boundary, homeomorphism, the classification problem etc. Also the Euler-Poincaré characteristic for polyhedra (or equivalently for a polygonal subdivision of a map of a surface) is defined: $\chi = V - E + F$ where V is the number of vertices, E the number of edges, and F the number of faces. This is an invariant, i.e., independent of the number and form of the patches used to map the surface. It only depends on the topology of the surface. A sphere has $\chi = 2$, while a torus has $\chi = 0$. The *genus* g of an orientable surface is defined by $\chi = 2 - 2g$. Thus a torus has genus 1, a sphere genus 0. A topologist is sometimes characterized as a mathematician not knowing the difference between a donut and a coffee cup since one may indeed deform a donut into a mug, the hole in the torus becoming the hole making it possible the grab the mug with its handle. In this sense, the genus of a surface is the number of handles or holes it has. Surfaces of genus $g \geq 2$ are the result of connecting g tori. And this is all there is for closed surfaces: a sphere ($g = 0$), a torus ($g = 1$) or connected tori ($g \geq 2$) according to the classification theorem.



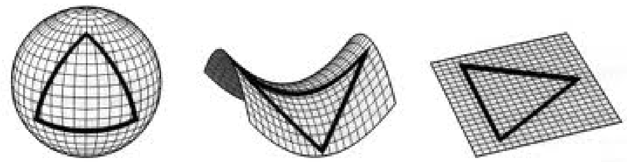
torus


bottle of Klein


donut → coffee cup


genus 3

Chapter 4 is about geometry of *Flatland*. What kind of geometries are possible in *Flatland* being unbounded, compact and orientable? If the postulate of parallel lines is accepted, then it follows as a consequence that the sum of the angles in a triangle is 180°. Thus if *Carrée* does not measure this, then, Euclidean geometry is excluded and it means that *Flatland* is not a plane. If *Flatland* is a sphere, then the sum of the angles of a triangle is 180° plus c times its area. The value of c is related to curvature. In a plane, this is zero, on a sphere it is positive and constant, but on more general surfaces like a hyperboloid or a torus, this will depend on the local shape at the position of the triangle, or one may define and average curvature over the surface.



Positive Curvature Negative Curvature Flat Curvature

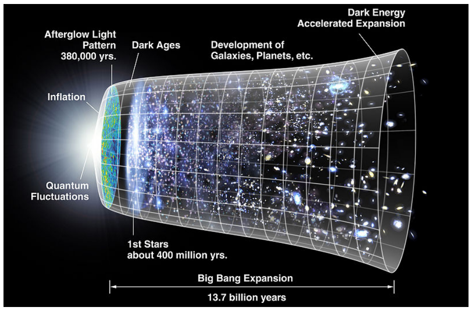
A distinction has to be made between intrinsic and extrinsic geometry. In *extrinsic geometry* curves on 2D surfaces are studied as curves in the 3D Euclidean space. But what can *Losange* (i.e., Diamond, the geometer of *Flatland*) do who is living on the surface, unaware of the ambient space, hence certainly of its geometry, that contains the surface he is living on? He is restricted to *intrinsic geometry* where he can measure angles, and distances, and draw straight lines (a straight line segment is the shortest path between two points), and draw circles (all points at a constant distance from its center). Differential geometry learns that for each point on a 2D surface we can find two orthogonal principal directions with curvature k_1 and k_2 respectively. Setting $K = k_1 k_2$, then this is related to the constant c mentioned above in the formula for the sum of the angles of a triangle. This is actually the *Gauss-Bonnet theorem*: $c = 180K/\pi$ (the $180/\pi$ factor transforms degrees to radians), and this is not only true for triangles. Summing over all possible triangulations (or subdivisions) of the surface it follows that 2π times the Euler-Poincaré characteristic equals the average curvature times the area of the whole surface. Hence the Gauss-Bonnet theorem links local with global geometry. If *Losange* cannot or does not want to measure at all points of his world, it is important to assume that his world is homogeneous and isotropic (isotropic implies homogeneous). That means that translation and rotation does not change the geometrical properties. With this hypothesis, *Losange* may assume that the curvature K in his world is constant. So he has as a consequence of this theory 3 possibilities: (1) $K > 0$ and $\chi > 0$: then he lives on a sphere (elliptic geometry), (2) $K = 0$ and $\chi = 0$, then his world is a torus, and (3) $K < 0$ and $\chi = 2 - 2g < 0$ then he lives on a surface of genus g (with hyperbolic geometry).

Now that *Flatland* is completely explored, by analogy, Muñoz can move on to three-dimensional geometry and the shape of our universe. The world is now a 3-variety or manifold and people living there can only make local observations. Unlike the case of *Flatland* there is not a classification of all 3-varieties yet. For example studying the simplest one (a 3-ball) was the subject of the Poincaré conjecture: every simply connected 3-variety without border is homeomorphic to a 3-sphere. It was formulated in 1904 but only solved by Perelman in 2003.

G. Perelman

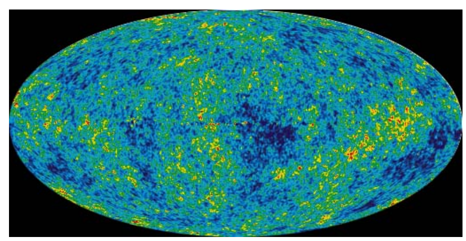
Because we cannot visualize things as we did before, we have, just as the compatriots of *Carrée*, to rely on an atlas. The pages of our atlas can be thought of as cubes of which the faces should be glued to faces of other pages. The ways in which this is done defines the shape of the world (sphere, torus, etc.) with possible alien side-effects. A torus is the connection of 2 spheres. Both the cup and the handle are homeomorphic to a sphere. They are connected by cutting 2 holes in each and glueing the holes of the cup to the holes of the handle. For people thinking that they live on the surface of a sphere (cup), these glueing sections appear to be 'stargates' where they can leave their sphere, to re-appear via a parallel sphere (handle) at another place on their sphere (cup). They will not know when they cross a 'gate' because they never leave the surface of the torus which they mistakingly think of as a sphere. Similarly, one may imagine 'stargates' connecting 3-spheres which would allow to take a 'shortcut' via a parallel world. But there are many more 3-varieties and many more possibilities to connect them, which could result in tori that interlace like knots. Even with the hypotheses of homogeneity and isotropy, (assuming constant curvature $K = k_1 k_2 k_3$), one can imagine different possible geometries. There are of course the isotropic geometries where the signs of k_1, k_2, and k_3 are the same. For example when they are all zero, it is Euclidean. But there are five other possible homogeneous geometries like $S^2 \times \mathbb{R}$ i.e., $k_1, k_2 > 0$ and $k_3 = 0$ giving rise to an elliptic geometry in 2 dimensions and Euclidean in the third dimension etc. Not all possibilities have been explored yet, but the reader may have an idea of how to generalize the strategies developed by *Carrée* and *Losange*.

The last chapter is then considering the question: What is the shape of our universe? It will be clear from what has preceded that there is not a definitive answer to this question yet. Aristotle thought it was a giant ball with the stars living on the spherical boundary. Riemann was the first to propose a non-Euclidean geometry for our universe, forcing to consider it as a variety. Einstein introduced the space-time variety of dimension 4 but here we are only looking at the space component. It makes sense and is generally accepted that the universe has no border, is homogeneous, isotropic, and orientable. It being compact is still a point of discussion.

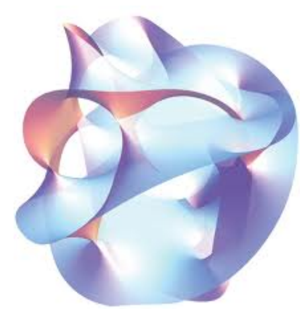


expansion of universe

Cosmological observations give information about the geometry of the universe. In the Friedmann-Lemaître-Robinson-Walker (FLRW) model of an expanding universe (F) after the Big Bang (L) it is assumed that the curvature K is constant and therefore our universe is either a 3-sphere ($K > 0$), a Euclidean space ($K = 0$) or it is hyperbolic ($K < 0$). Compactness is guaranteed in the first case but it not in the other two. Since mass changes the curvature K, the future of the universe will depend on the mass distribution ρ, which changes during the expansion. There is a critical density ρ_C for which $K = 0$. Thus if $\rho > \rho_C$ then the expansion of the universe will stop and shrink again until the *Big Crunch*. If $\rho < \rho_C$ expansion will go on until it will end in a *Big Rip* by loss of gravitational cohesion or in a *Big Freeze* by loss of free energy. In the limiting third case $\rho = \rho_C$ the expansion will end in some kind of stable situation. So in this model it is important to know the density of matter. This may be relatively simple for stars, but it is not so simple for dark matter, black holes, intergalactic dust, etc. Nowadays there are alternatives for the FLRW model.


CMB map observed by WMAP satellite

The space-time universe where $K \leq 0$ will not be compact, but this does not exclude that the space component is compact. If the space geometry is hyperbolic, then there are many more 3-varieties possible that describe the topology than in the other cases. Hence one may reason that the chance of an hyperbolic space component is larger but confirmation that this is effectively the case by experimental observations is as yet not given. Our observation are always local and confined to the *last scattering surface* (LSS). Accepting that the universe is 13,76 billion years, and given the speed of light, but taking into account the expansion of the universe, this means that the the LSS is a ball of diameter 93 billion light-years and growing. If the universe is larger, there is no hope that we shall ever be able to determine its global topology in this way. We observe this LSS by measuring the *Cosmic Microwave Background* (CMB), i.e., the afterglow of the Big Bang. Observations of the *Wilkinson Microwave Anisotropy Probe* (WMAP) by NASA and the *Planck* project by ESA seem to be in correspondence with an expanding flat homogeneous universe but others have a different interpretation.


3D projection of 5D
Calabi-Yau manifold

Vicente Muñoz has done an excellent job in explaining the difference between geometry and topology for the layman. Although the mathematical technicalities of these disciplines are quiet complicated, all of this is well hidden in this account about what these disciplines can learn us and what are still open problems in trying to give answers to fundamental questions related the shape of our universe. To go beyond space, and understand the geometrical world of strings, the reader should shift to a much higher gear. This book will not bring you anywhere near the geometries of Calabi-Yau manifolds, the birth of which are described in *The Shape of Inner Space, String Theory and the Geometry of the Universe's Hidden Dimensions* (S.-T. Yau & S. Nadis, Basic Books, 2010). Adhemar Bultheel